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Painlevé analysis for the mixmaster universe model

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Abstract. We show that the mixmaster universe, or Bianchi IX, model passes the Painlevé test in the form of the Ablowitz–Ramani–Segur algorithm, i.e. the solutions of the equations of motion do not have movable critical points. Thus this system is probably integrable and therefore non-chaotic.

1. Introduction

The mixmaster universe model was introduced by Belinski and Khalatnikov [1] (and independently by Misner [2]) and has been studied extensively over the years. This model is homogeneous but anisotropic: it expands along two directions and contracts along the third one. As one approaches the initial singularity, backwards in time, the directions of expansion and contraction change, presumably in a chaotic way, an infinite number of times. This system was believed to be ergodic and mixing [3, 4], i.e. maximally chaotic.

In order to check the chaotic character of the mixmaster universe, several people calculated the maximal Liapunov characteristic number (LCN). The first calculations [5, 6] found that the maximal LCN is positive, thus indicating chaos, but more accurate calculations [7–10] have shown that the maximal LCN is zero. Thus it seems that the mixmaster universe is not chaotic. Recently a number of attempts have been made to redefine the LCNs or find other indicators of chaos, in order to save the chaotic behaviour of the mixmaster universe. However, these have not produced any definite results.

In order to avoid these ambiguities we investigate here the possible integrability of the mixmaster universe model by a completely different method, namely singularity analysis, a well-known integrability detector. Using singularity analysis we check whether this model has the Painlevé property, i.e. whether its solutions have no critical singularities. In order to perform the analysis we write the model in Hamiltonian form, apply the Ablowitz–Ramani–Segur (ARS) [11] algorithm and show that the mixmaster universe passes the Painlevé test. This fact is a strong indication [12, 13] that this model is probably integrable.

2. The equations of motion

The solution of Einstein's equations in the case of the mixmaster universe, or Bianchi IX, model, are written [14]:

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$$2\ddot{\alpha} = (e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha}$$
$$2\ddot{\beta} = (e^{2\gamma} - e^{2\alpha})^2 - e^{4\beta} \tag{1}$$
$$2\ddot{\gamma} = (e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma}$$

and we have

$$4(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\gamma} + \dot{\gamma}\dot{\alpha}) = e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2e^{2(\alpha+\beta)} - 2e^{2(\beta+\gamma)} - 2e^{2(\gamma+\alpha)} \tag{2}$$

where dots mean derivatives with respect to the logarithmic time τ which is related to the coordinate time t by the relation

$$\tau = -\ln t. \tag{3}$$

If we introduce the variables

$$x = 2\alpha \qquad y = 2\beta \qquad z = 2\gamma$$
$$p_x = -(\dot{y} + \dot{z}) \qquad p_y = -(\dot{z} + \dot{x}) \qquad p_z = -(\dot{x} + \dot{y}) \tag{4}$$

we find

$$2\dot{x} = p_x - p_y - p_z \qquad 2\dot{y} = p_y - p_z - p_x \qquad 2\dot{z} = p_z - p_x - p_y \tag{5}$$
$$\dot{p}_x = 2e^x(e^y + e^z - e^x) \qquad \dot{p}_y = 2e^y(e^z + e^x - e^y) \qquad \dot{p}_z = 2e^z(e^x + e^y - e^z) \tag{6}$$

with

$$H \equiv \tfrac{1}{4}(p_x^2+p_y^2+p_z^2-2p_xp_y-2p_yp_z-2p_zp_x)+e^{2x}+e^{2y}+e^{2z}-2e^{x+y}-2e^{y+z}-2e^{z+x} = 0. \tag{7}$$

The variables x, y, z, p_x, p_y, p_z are canonical variables in the Hamiltionian H and the energy has the particular value zero.

3. The Painlevé analysis

In order to perform the singularity analysis we transform the system (5), (6) to one involving rational expressions only. We introduce the new (non-canonical) variables

$$X = e^x \qquad Y = e^y \qquad Z = e^z \tag{8}$$

and rewrite the equations of motion:

$$2\dot{X} = X(p_x - p_y - p_z) \qquad 2\dot{Y} = Y(p_y - p_z - p_x) \qquad 2\dot{Z} = Z(p_z - p_x - p_y) \tag{9}$$
$$\dot{p}_x = 2X(Y + Z - X) \qquad \dot{p}_y = 2Y(Z + X - Y) \qquad \dot{p}_z = 2Z(X + Y - Z). \tag{10}$$

We apply now the ARS [11] algorithm, to equations (9) and (10). We recall that this algorithm has three steps:

(i) Obtain the leading singular behaviours. This means that one must determine all the possible degrees of virtual poles of solutions as well as the leading coefficients.

(ii) Look for the resonances. This is done by expanding the solutions in a Laurent series and by obtaining recurrence relations for the coefficients: the resonances correspond to the cases in which one of the coefficients is not uniquely determined.

(iii) Check the compatibility at the resonances. In the resonant case one obtains an inhomogeneous linear system. The compatibility condition ensures the solvability of this system despite the nullity of the determinant.

If $\tau = \tau_0$ is a pole of the solution of these equations we first find the leading terms by setting

$$\begin{aligned} &X = x_1 s^{m_1} \qquad Y = x_2 s^{m_2} \qquad Z = x_3 s^{m_3} \\ &p_x = p_1 s^{n_1} \qquad p_y = p_2 s^{n_2} \qquad p_z = p_3 s^{n_3} \end{aligned} \tag{12}$$

(where $s = \tau - \tau_0$) in (9) and (10):

$$\begin{aligned} m_1 x_1 s^{m_1-1} &= \tfrac{1}{2} x_1 s^{m_1}(p_1 s^{n_1} - p_2 s^{n_2} - p_3 s^{n_3}) \\ m_2 x_2 s^{m_2-1} &= \tfrac{1}{2} x_2 s^{m_2}(p_2 s^{n_2} - p_3 s^{n_3} - p_1 s^{n_1}) \\ m_3 x_3 s^{m_3-1} &= \tfrac{1}{2} x_3 s^{m_3}(p_3 s^{n_3} - p_1 s^{n_1} - p_2 s^{n_2}) \end{aligned} \tag{13}$$

$$\begin{aligned} n_1 p_1 s^{n_1-1} &= 2 x_1 s^{m_1}(x_2 s^{m_2} + x_3 s^{m_3} - x_1 s^{m_1}) \\ n_2 p_2 s^{n_2-1} &= 2 x_2 s^{m_2}(x_3 s^{m_3} + x_1 s^{m_1} - x_2 s^{m_2}) \\ n_3 p_3 s^{n_3-1} &= 2 x_3 s^{m_3}(x_1 s^{m_1} + x_2 s^{m_2} - x_3 s^{m_3}). \end{aligned} \tag{14}$$

Without loss of generality we can assume $m_1 \leqslant m_2 \leqslant m_3$, $n_1 \leqslant n_2 \leqslant n_3$. We distinguish two main cases:

Case I: $n_1 < n_2 \leqslant n_3$. As we shall see, this is generic singular behaviour with six free parameters. The lowest possible exponents in equations (13) and (14) give

$$n_1 = -1 \qquad n_2 = 0 \qquad n_3 = 0 \tag{15}$$

and

$$m_1 = -1 \qquad m_2 = 1 \qquad m_3 = 1. \tag{16}$$

The coefficients of the lowest-order terms in (13) and (14) give

$$p_1 = -2 \qquad x_1 = \pm \mathrm{i} \tag{17}$$

while x_2, x_3, p_2, p_3 remain free. We now look for the resonances by setting

$$\begin{aligned} &X = x_1 s^{m_1} + \gamma_1 s^{m_1+r} \qquad Y = x_2 s^{m_2} + \gamma_2 s^{m_2+r} \qquad Z = x_3 s^{m_3} + \gamma_3 s^{m_3+r} \\ &p_x = p_1 s^{n_1} + \delta_1 s^{n_1+r} \qquad p_y = p_2 s^{n_2} + \delta_2 s^{n_2+r} \qquad p_z = p_3 s^{n_3} + \delta_3 s^{n_3+r} \end{aligned} \tag{18}$$

in equations (9) and (10). The coefficients of the terms linear in (γ_i, δ_i) give

$$\gamma_1(r-1) = -\gamma_1 + \tfrac{1}{2} x_1 \delta_1 \qquad \gamma_2(r+1) = \gamma_2 - \tfrac{1}{2} x_2 \delta_1 \qquad \gamma_3(r+1) = \gamma_3 - \tfrac{1}{2} x_3 \delta_1 \tag{19}$$

$$\delta_1(r-1) = -4 x_1 \gamma_1 \qquad \delta_2 r = 0 \qquad \delta_3 r = 0 \tag{20}$$

and the resonances are obtained by setting the determinant of the coefficients of (γ_i, δ_i) to zero. We find

$$(r+1)r^4(r-2) = 0. \tag{21}$$

The resonance -1 is related, as usual, to the freedom of the location τ_0 of the singularity, while the quadruple 0 resonance is related to the free x_2, x_3, p_2, p_3 parameters. It thus remains to investigate whether the $r = 2$ resonance satisfies the compatibility condition by expanding all variables up to terms linear in s and substituting in the full system (9), (10). Thus we check whether or not logarithmic terms enter the singular expansion. It turns out that the resonance $r = 2$ does not lead to incompatibilities and we find the following singular expansion:

$$X = \pm\frac{\mathrm{i}}{s} + 0 + \gamma_1 s + \cdots \qquad Y = x_2 s + \cdots \qquad Z = x_3 s + \cdots \tag{22}$$

$$p_x = -\frac{2}{s} + (p_2 + p_3) \pm 2\mathrm{i}(x_2 + x_3 - 2\gamma_1)s + \cdots \qquad p_y = p_2 \pm 2\mathrm{i}x_2 s + \cdots$$
$$p_z = p_3 \pm 2\mathrm{i}x_3 s + \cdots$$

where the free parameter γ_1 enters at $r = 2$. Thus this expansion is generic, i.e. it has six free parameters, and it is of Painlevé type. (Special cases of the dominant singular expansion can be obtained by putting one or both of the p_2, p_3 to zero.)

Next we must examine non-generic expansions. One can easily convince oneself that no singular solution with only *two* divergent (x_i, p_i) can exist. We thus turn now to the second case:

Case II: $n_1 = n_2 = n_3$. The same analysis as in the generic case leads to

$$\begin{aligned} &n_1 = n_2 = n_3 = -1 \qquad && m_1 = m_2 = m_3 = -1 \\ &x_1 = x_2 = x_3 = \pm\mathrm{i} && p_1 = p_2 = p_3 = 2. \end{aligned} \tag{23}$$

The resonances are straightforward to compute and we find

$$(r+1)^3(r-2)^3 = 0. \tag{24}$$

The appearance of the triple (-1) resonance may look puzzling at first sight. However, there is nothing wrong with it: one of the (-1) is associated with the freedom of τ_0, while the remaining two (-1)s indicate that the positions of the singularities of all three variables X, Y and Z were initially free and by choosing the behaviour $1/s$ we forced all of them to diverge at the same point τ_0. No logarithmic terms enter at this resonance. On the other hand, the triple 2 is something that may lead to incompatibilities. Expanding all x_i and p_i we have checked that compatibility at $r = 2$ is indeed satisfied and thus this non-generic expansion

$$X = \pm\frac{\mathrm{i}}{s} + \gamma_1 s + \cdots \qquad Y = \pm\frac{\mathrm{i}}{s} + \gamma_2 s + \cdots \qquad Z = \pm\frac{\mathrm{i}}{s} + \gamma_3 s + \cdots \tag{25}$$

$$p_x = \frac{2}{s} \pm 2\mathrm{i}(\gamma_2+\gamma_3)s + \cdots \qquad p_y = \frac{2}{s} \pm 2\mathrm{i}(\gamma_3+\gamma_1)s + \cdots \qquad p_z = \frac{2}{s} \pm 2\mathrm{i}(\gamma_1+\gamma_2)s + \cdots$$

is also of Painlevé type depending on four free parameters. In conclusion, the mixmaster universe model satisfies the Painlevé criterion of integrability.

4. Conclusion

The mixmaster universe was proposed as a chaotic model of the early universe. Its chaotic character was assessed through qualitative analyses and numerical calculations. However, a detailed computation of the maximal LCN did not lead to positive values (which is an indicator of chaos), using either the τ-time of Belinski *et al* [1] or the Ω-time (which is equal to $\Omega = \frac{1}{6}(x+y+z)$) of Arnowitt *et al* [15]. Thus certain authors [16–18] introduced new time variables in order to derive a positive LCN. However, such a method is ambiguous. In fact, even an integrable system, which is known to be non-chaotic, can give a positive LCN by an appropriate time transformation. For example, a linear deviation of two nearby orbits $\chi = \chi_0 t_1$ that gives zero LCN becomes an exponential deviation in another time t_2 related to t_1 by the relation $t_1 = e^{qt_2}$ with $q > 0$: in time t_2 the LCN is positive. In this paper we tested the possible integrability of the mixmaster model using singularity analysis and we found that the equations of motion pass the Painlevé test. Thus the system is probably integrable [12, 13]. In order to have an independent check, we also performed an analysis based on Ziglin's theorem [19] using the methods of [20]. We found that the system satisfies, in a non-trivial way, the condition required by this theorem for integrability. So Ziglin's approach does not prove non-integrability of the system at hand. Still, the final proof of integrability would be the computation of two constants of motion (besides the Hamiltonian). However, our preliminary calculations did not lead to any positive result for integrals polynomial in the momenta of low degree. Clearly the system deserves further investigation.

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